a theoretical prediction of the density gradient also is shown. This theoretical result was obtained by numerically integrating the Bethe-Teller vibrational relaxation equation subject to the shock-wave conservation relations and with the relaxation-time expression suggested by White and Millikan.[8] The agreement between experiment and theory is quite good, the maximum discrepancy being about equal to the experimental uncertainty (associated principally with the uncertainties in system calibration and in the recording and transfer of data using oscillograms). As suggested by Kiefer et al., zero time was taken at the peak in the voltage signal, in this case at 0.5 μsec after the initial transient. An effective lower limit on the spatial resolution near the shock front for this experiment is therefore about 0.9 mm. Experiments at higher pressure with less shock-wave curvature would permit resolution approaching the halfwidth of the laser beam.

Experimental results for the end-wall pressure, obtained with a Baganoff capacitance-type pressure gauge,[9] are presented in normalized form in Fig. 3. Also shown are two theoretical wall-pressure time histories calculated using the simple relation for pressure and density under test here. The solid curve is based on the density profile $\eta(s)$ predicted by White and Millikan's expression for relaxation time ($P\tau$), and the cross (+) points were computed with the density profile obtained by integrating the laser-schlieren data shown in Fig. 2. The density gradient was integrated backward in time from vibrational equilibrium, thereby yielding $\eta(s)$ since η_{final} is known from the initial conditions and the measured shock speed.

The prescription for mapping the distance s into the time after shock reflection t, derivable from simple geometrical arguments,[1,3] is

$$t = (s/V_s)[(1 + V_r/a_5)/(1 + V_r/V_s)]$$

where V_s and V_r are, respectively, the incident- and reflected-shock speeds, and a_5 is the equilibrium speed of sound in the reflected-shock region. Both V_r and a_5 are evaluated using the Rankine-Hugoniot shock-jump relations assuming vibrational equilibrium upstream and downstream of the reflected shock wave.[1]

The good agreement between measured and calculated pressure histories is taken as verification, within the experimental undertainty, of the simple relationship between pressure and density. (It is worth noting that the slight decrement in measured pressure visible for small t is expected due to the known effect of heat conduction to the shock-tube end wall.[10]) The close agreement between the two theoretical pressure-time histories follows from the agreement previously shown in Fig. 2, since the same pressure-density relationship was applied in both cases.

The present study, of course, also serves to further substantiate the relaxation-time expression for O_2 suggested by White and Millikan.

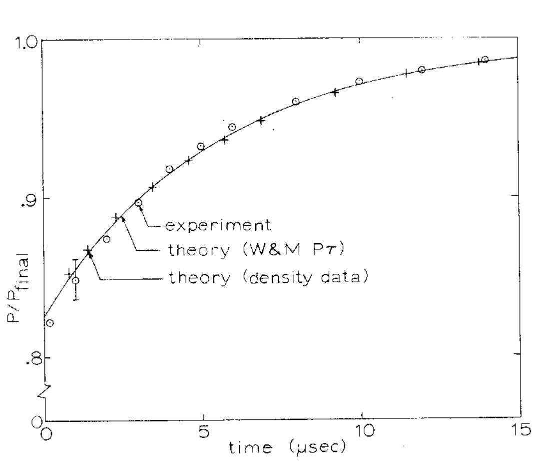

Fig. 3 Record of the end-wall pressure history for a shock wave in pure O_2. The experimental conditions were: $P_1 = 4.33$ torr, $V_s = 1.817$ mm/μsec, $T_1 = 299$°K. The theoretical results were calculated using the simple theory relating pressure and density for shock-wave reflection: —— based on the relaxation time expression of White and Millikan[8]; +, based on an integration of the laser-schlieren data in Fig. 2.

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Integral Equation for Small Perturbations of Irrotational Flows

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Introduction

THE problem of computing the effects of thickness on aerodynamic forces acting on oscillating wings and bodies, especially in the transonic speed range, has attracted considerable interest. The aim of this Note is to explore the validity of analytical methods which are based on deriving an integral equation from the application of small perturbations to a nonuniform but irrotational flow, such as might approximate the flow over a thick wing or body. Thus, the following observations apply to a wide Mach number range, but are restricted to small amplitude motions, such as at the onset of flutter, and to nonviscous flows.

It is shown that the integral equation relating the unknown velocity potential to the known normal flow velocity can be derived from the appropriate Green's identity. However, the Green's function from which the kernel of this integral equation is formed is not the unit source solution, but the unit solution

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of the adjoint wave equation. This distinction disappears in the uniform flow case, so that methods based on distributions of doublets are correct for uniform flows. However, when the flow is not uniform, the kernel of the integral equation should be formulated from solutions of the adjoint equation. Present methods of analysis for uniform flows, such as those of Watkins et al.[1] and Albano and Rodden[2] are formally based on doublet distributions. Should they be adapted to nonuniform flows, it would first be necessary to reformulate them in terms of the correct Green's function.

Derivation of the Wave Operator

The general equation for an irrotational flow in an inviscid, nonconducting medium, is derived readily from the continuity equation, and from Euler's equation. It is given by Garrick[3] as

$$\partial^2\Phi/\partial t^2 + \partial q^2/\partial t + \tfrac{1}{2}\mathbf{q}\cdot\nabla q^2 = c^2\nabla^2\Phi \tag{1}$$

where $\mathbf{q}$ is the flow velocity vector derived from a potential Φ by the equation

$$\mathbf{q} = \nabla\Phi \tag{2}$$

and c is the local velocity of sound, defined by

$$c^2 = dP/d\rho \tag{3}$$

where P is the pressure, and ρ the density.

The relationship between the flow variables and the pressure P is given by Bernouilli's equation

$$\partial\Phi/\partial t + \tfrac{1}{2}q^2 + \int dP/\rho = f(t) \tag{4}$$

It will now be assumed that the flow is composed of a steady component, representative of the flow around a wing or body, and of a small harmonic perturbation of angular frequency ω, so that

$$\Phi = \Phi_0 + \phi\, e^{i\omega t} \tag{5}$$

and

$$\mathbf{q} = \mathbf{V} + \nabla\phi\, e^{i\omega t} \tag{6}$$

where $\mathbf{V}$, which can also be expressed as $\nabla\Phi_0$, is the local steady flow velocity, and ϕ is the velocity potential of the perturbation.

If a solution can be obtained for the perturbation velocity potential ϕ, then Bernouilli's equation, Eq. (4), can be used to find the corresponding harmonic component of pressure, and the objectives will have been achieved. The remaining discussion will be restricted to the problem of finding the velocity potential ϕ.

Application of Eqs. (5) and (6) to Eq. (1) with appropriate elimination of small quantities of second order yields the linear equation

$$\mathscr{L}\{\phi\} = 0 \tag{7}$$

where the linear wave operator $\mathscr{L}$ is defined by

$$\mathscr{L} = \nabla^2 + \frac{\omega^2}{c^2} - \frac{2i\omega}{c^2}\mathbf{V}\cdot\nabla - \frac{1}{c^2}(\mathbf{V}\cdot\nabla)^2 - \frac{1}{2c^2}[\nabla V^2]\cdot\nabla \tag{8}$$

The square bracket is intended to signify that the influence of the operator contained in it is restricted. Thus $[\nabla V^2]$ is a vector which does not operate differentially on the terms which follow it, while $(\mathbf{V}\cdot\nabla)$ is a scalar operator.

Formal Procedure for Deriving Green's Equation

It is necessary to distinguish between two vector coordinates, $\mathbf{r}$, with components x, y, z, and $\boldsymbol{\rho}$, with components ξ, η, ζ. Then the operator $\nabla_{\mathbf{r}}$ has components $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, while the operator ∇_ρ has components $\partial/\partial\xi$, etc., so that Eq. (7) can be written in two ways

$$\mathscr{L}_{\mathbf{r}}\{\phi(\mathbf{r})\} = 0 \tag{9a}$$

$$\mathscr{L}_\rho\{\phi(\boldsymbol{\rho})\} = 0 \tag{9b}$$

while the equation for a source located at the point $\boldsymbol{\rho}$ would be

$$\mathscr{L}_{\mathbf{r}}\{\phi_s(\mathbf{r},\boldsymbol{\rho})\} = \delta(\mathbf{r}-\boldsymbol{\rho}) \tag{10}$$

where ϕ_s is the unit source potential function. In order to proceed with the derivation of Green's equation, it is necessary to define a function G which satisfies the equations

$$\tilde{\mathscr{L}}_{\mathbf{r}}\{G(\boldsymbol{\rho},\mathbf{r})\} = \delta(\mathbf{r}-\boldsymbol{\rho}) \tag{11a}$$

$$\tilde{\mathscr{L}}_\rho\{G(\mathbf{r},\boldsymbol{\rho})\} = \delta(\mathbf{r}-\boldsymbol{\rho}) \tag{11b}$$

where $\tilde{\mathscr{L}}$ is the adjoint wave operator. The function G is variously referred to in the literature as the free-field Green's function, the unit solution, or when the operator is hyperbolic, the Reimann function. It will be referred to here as the Green's function. The distinction between the functions ϕ_s and G is of considerable importance, and will be discussed later. The adjoint wave operator $\tilde{\mathscr{L}}$ can be found by expressing the wave operator $\mathscr{L}$ in the form of Green's identity

$$G\mathscr{L}\{\phi\} = \phi\tilde{\mathscr{L}}\{G\} + \nabla\cdot\mathbf{P}\{G;\phi\} \tag{12}$$

where $\mathbf{P}$ is the bilinear concomitant. After rearranging this, and integrating over the complete space $\mathscr{v}$

$$\int_{\mathscr{v}} \phi(\boldsymbol{\rho})\tilde{\mathscr{L}}_\rho\{G(\mathbf{r},\boldsymbol{\rho})\}\, d\mathscr{v} = \int_{\mathscr{v}} G(\mathbf{r},\boldsymbol{\rho})\mathscr{L}_\rho\{\phi(\boldsymbol{\rho})\}\, d\mathscr{v} - \int_{\mathscr{v}} \nabla_\rho\cdot\mathbf{P}\{G(\mathbf{r},\boldsymbol{\rho});\ \phi(\boldsymbol{\rho})\}\, d\mathscr{v} \tag{13}$$

and, on substitution from Eqs. (9b) and (11b), together with application of the divergence theorem

$$\phi(\mathbf{r}) = -\int_{\mathscr{s}} \hat{\boldsymbol{\mu}}\cdot\mathbf{P}\{G(\mathbf{r},\boldsymbol{\rho});\ \phi(\boldsymbol{\rho})\}\, d\mathscr{s} \tag{14}$$

where $\hat{\boldsymbol{\mu}}$ is the unit outward normal on the surface $\mathscr{s}$ bounding the space $\mathscr{v}$.

Equation for the General Flow Case

On substitution of the wave operator given in Eq. (8), Eq. (12) takes the form

$$\begin{aligned}
G\left\{\nabla^2 + \frac{\omega^2}{c^2} - \frac{2i\omega}{c^2}\mathbf{V}\cdot\nabla - \frac{1}{c^2}(\mathbf{V}\cdot\nabla)^2 - \frac{1}{2c^2}[\nabla V^2]\cdot\nabla\right\}\phi = \\
\phi\left\{\nabla^2 + \frac{\omega^2}{c^2} + 2i\omega\nabla\cdot\frac{\mathbf{V}}{c^2} - (\nabla\cdot\mathbf{V})^2\frac{1}{c^2} + \nabla\cdot\frac{1}{2c^2}[\nabla V^2]\right\}G + \\
\nabla\cdot\left\{G\nabla\phi - \phi\nabla G - \frac{2i\omega}{c^2}\mathbf{V}G\phi - \frac{\mathbf{V}}{c^2}(\mathbf{V}\cdot\nabla\phi) + \right. \\
\left. \mathbf{V}\phi\left(\nabla\cdot\frac{\mathbf{V}}{c^2}\right)G - \frac{1}{2c^2}[\nabla V^2]G\phi\right\}
\end{aligned} \tag{15}$$

The term in the first set of brackets on the right-hand side is clearly the adjoint operator

$$\tilde{\mathscr{L}} = \nabla^2 + \frac{\omega^2}{c^2} + 2i\omega\nabla\cdot\frac{\mathbf{V}}{c^2} - (\nabla\cdot\mathbf{V})^2\frac{1}{c^2} + \nabla\cdot\frac{1}{2c^2}[\nabla V^2] \tag{16}$$

while the term in the second set is the bilinear concomitant $\mathbf{P}$. This can be substituted into Eq. (14), resulting in

$$\phi(\mathbf{r}) = \int_{\mathscr{s}} \left\{\phi(\boldsymbol{\rho})\frac{\partial G(\mathbf{r},\boldsymbol{\rho})}{\partial\mu} - G(\mathbf{r},\boldsymbol{\rho})\frac{\partial\phi(\boldsymbol{\rho})}{\partial\mu}\right\} d\mathscr{s} \tag{17}$$

where μ is the local normal coordinate to the surface $\mathscr{s}$, so that

$$\hat{\boldsymbol{\mu}}\cdot\nabla_\rho = \partial/\partial\mu \tag{18}$$

In deriving Eq. (17), the boundary conditions on the steady flow component were applied in the form

$$\hat{\boldsymbol{\mu}}\cdot\mathbf{V} = 0 \tag{19}$$

which requires that there be no flow component normal to the surface $\mathscr{s}$.

In the most common form of the problem statement, the unsteady pressures are to be found, given the normal components of unsteady flow velocity on the surface. The surface $\mathscr{s}$ is generally internal to the space $\mathscr{v}$, which extends to infinity. Because of a shear across the wake surface, behind a wing, it sustains a discontinuity in velocity potential, but not in pressure. It is therefore convenient to treat the wake as part of the surface, so that the surface $\mathscr{s}$ includes not only the complete wing and body surfaces but also both sides of the wake.

The flow velocity in any direction defined by the normal coordinate n is obtained by taking $\partial/\partial n$ of Eq. (17), resulting in

$$\partial\phi(\mathbf{r})/\partial n+\int_{\mathscr{s}} [\partial\phi(\boldsymbol{\rho})/\partial\mu][\partial G(\mathbf{r},\boldsymbol{\rho})/\partial n]\, d\mathscr{s} = \int_{\mathscr{s}} \phi(\boldsymbol{\rho})[\partial^2 G(\mathbf{r},\boldsymbol{\rho})/\partial n\,\partial\mu]\, d\mathscr{s} \qquad (20)$$

Specialization to Uniform Flow Case

When the flow is not transonic, a thin airfoil can be treated by assuming c and $\mathbf{V}$ to be uniform everywhere, so that the two-sided surface $\mathscr{s}$ can be replaced by a single-sided surface s, having a local unit normal v. Then, in the absence of "breathing" modes of the airfoil, the integral on the left-hand side of Eq. (20) cancels out, while in the integral on the right-hand side, the potentials can be replaced by their difference $\Delta\phi$. This is taken as positive when ϕ decreases in the positive v direction. Equation (20) now becomes

$$\frac{\partial\phi(\mathbf{r})}{\partial n} = \int_s \Delta\phi(\mathbf{r})\frac{\partial^2 G(\mathbf{r},\boldsymbol{\rho})}{\partial n\,\partial v}\, ds \qquad (21)$$

Reversibility Relations in the Uniform Flow Case

When $\mathbf{V}$ and c are everywhere uniform, Eqs. (8) and (16) for the wave operator and its adjoint can be reduced to

$$\mathscr{L} = \nabla^2 - [(i\omega/c)-(\mathbf{V}/c)\cdot\nabla]^2 \qquad (22)$$

$$\tilde{\mathscr{L}} = \nabla^2 - [(i\omega/c)+(\mathbf{V}/c)\cdot\nabla]^2 \qquad (23)$$

Clearly

$$\tilde{\mathscr{L}} = \mathscr{L}^* \qquad (24)$$

where $\mathscr{L}^*$ is the adjoint of $\mathscr{L}$, so that the operator is Hermetian. From another point of view, $\tilde{\mathscr{L}}$ is the wave operator for reversed flow. It can now be shown that

$$\tilde{\mathscr{L}}_{\rho}\{G(\mathbf{r},\boldsymbol{\rho})\} = \mathscr{L}_{\mathbf{r}}\{G(\mathbf{r},\boldsymbol{\rho})\} = \delta(\mathbf{r}-\boldsymbol{\rho}) \qquad (25)$$

By comparison with Eq. (10) it is readily seen that G is identical to the unit source potential function ϕ_s for a uniform flow case. This has been assumed by many investigators without benefit of proof.

Conclusions

In the general case of irrotational flow, ϕ_s and G, as given by Eqs. (10) and (11b), are distinct functions, which represent disturbances spreading in opposite directions. The unit source $\phi_s(\mathbf{r},\boldsymbol{\rho})$ represents a disturbance spreading from the point $\boldsymbol{\rho}$, whereas $G(\mathbf{r},\boldsymbol{\rho})$, which appears in Eq. (20), represents a disturbance spreading from the point $\mathbf{r}$, the point at which the normal flow $\partial\phi/\partial n$ is defined. Thus the present analysis appears to conform to a concept of cause and effect according to which the normal flow resulting from a deflection of the aerodynamic surfaces causes disturbances to spread and to induce perturbations of the velocity potentials and therefore of the pressures acting on the surface at the points $\boldsymbol{\rho}$. If the unit source ϕ_s were to appear in Eq. (20), it would be necessary to assume that the induced velocity potentials anticipated the deflections of the aerodynamic surfaces, in apparent contradiction of cause and effect. It is only in the uniform flow case that this distinction disappears.

In conclusion, it will be necessary to obtain solutions to Eq. (11b) using the adjoint wave operator, in order to set up the integral equation expressed by Eq. (20). For example, if ray tracing techniques should be used to develop expressions for the function G, these would not be acoustical rays obtained from the wave equation, but rather rays satisfying the adjoint wave equation. Also, the rays would start at the point $\mathbf{r}$, where the value for $\partial\phi/\partial n$ is given, and they would proceed to the point $\boldsymbol{\rho}$, where the value for the unknown velocity potential ϕ is sought. This is in the opposite direction to the rays implied by the unit source solution ϕ_s, which originate at the point $\boldsymbol{\rho}$.

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Thin-Walled Elements in Truss Synthesis

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Introduction

In order to obtain a definitive lower-bound to the weight of optimized truss structures it is necessary to consider possible Euler and local buckling of compression members. Since critical stress for each instability mode is generally a function of a different design variable, early formulations of this inequality-constrained synthesis problem involved at least two variables per element.[1] However, it was subsequently shown[2] that imposition of simultaneous buckling in Euler and local modes allowed the formulation to be reduced to one involving a single variable and single buckling constraint per element. More recently an iterative approach, rather than a mathematical programing solution, has been used to obtain truss designs which include consideration of element stability.[3,4] Although these designs are apparently obtained by treating member areas as the only variables, the formulation seems to utilize separate constraints on Euler and local buckling stresses, and is therefore not directly related to Ref. 2.

The purpose of this Note is to present a rigorous proof of the formulation contained in Ref. 2, wherein the effect of element instability is included in the truss synthesis problem in such a way that the only required design variables are cross-sectional areas. It is shown that this formulation also leads directly to an iterative procedure which allows simple and efficient design of near-optimal fully-stressed buckling-resistant trusses.

Thin-Walled Truss Components

Consider the design of a uniform thin-walled round tubular truss member. Behavioral constraints for this element are stated as follows:

$$-\sigma_c \leqq \sigma_{ij} \leqq \sigma_T \qquad (1a)$$

$$\sigma_{ij} \geqq -\pi^2 EI_i/(L_i{}^2 A_i) \qquad (1b)$$

$$\sigma_{ij} \geqq -KE(t/D)_i \qquad (1c)$$

where σ_{ij} is the stress in element i under load condition j, σ_T and σ_c are tensile and compressive yield stresses, respectively, L = length, E = modulus of elasticity, D = mean diameter, t = wall thickness, I = moment of inertia, A = cross-sectional area, and K is a buckling coefficient. It has been implied in Eqs. (1) that all elements are composed of the same material, although generalization can be accomplished in a straightforward manner, as can extension to elements of other cross-sectional shapes.

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